Marketing resource allocation in duopolies over social networks

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Traditional marketing

1. Broadcast via TV/Radio etc,
2. Exploit influence of celebrities
3. Not customized for viewers
SM platforms collect user data

2 Firms compete online to provide ads based on user data

3 Exploits individual preferences and opinions.
State of the art

- Marketing games classical literature [L. Friedman et al. 1958], [Butters et al. 1977],
- Targeted ads (second price auction game) [Edelman et al. 2007],
- Opinion dynamics aware targeted ads, but with all to all graph [Masucci et al. 2014].
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1. Study the strategies of competing firms marketing over social networks,
2. firms are aware of the social network graph and opinions.
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Action of firm \( i \in \{1, 2\} \) is \( a_i \in A_i \) and

\[
A_i := \left\{ a_i \in [0, b_i]^N \mid \sum_{n=1}^{N} a_{i,n} \leq B_i \right\} \quad (1)
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- Action of firm $i \in \{1, 2\}$ is $a_i \in \mathcal{A}_i$ and

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\]

- Agents interact over a directed and weighted graph $(\mathcal{V}, \mathcal{E}, \Omega)$, with $\Omega_{m,n}$ describing the influence of agent $m$ on $n$. 
The Laplacian of the graph is given by

\[
L_{m,n} = \begin{cases} 
\sum_{n=1}^{N} \Omega_{m,n} & \text{if } m = n \\
\Omega_{m,n} & \text{if } m \neq n 
\end{cases}.
\]  

(2)

Dynamics model

\[
\begin{align*}
\dot{x}(t) &= -Lx(t) \\
x_n(t^+_k) &= \phi(x_n(t_k), a_{1,n}, a_{2,n}) \\
\end{align*}
\]  

\forall t \in \mathbb{R} \setminus \{0\} \quad \forall n \in \mathcal{V}, t_k \in \mathcal{T}.

(3)
\[ \phi(x_{0,n}, a_{1,n}, a_{2,n}) = \frac{x_{0,n} + a_{1,n}}{1 + a_{1,n} + a_{2,n}}, \; \forall n \in \{1, \ldots, N\}. \]
 Opinion dynamics: at campaign

\[
\phi(x_{0,n}, a_{1,n}, a_{2,n}) = \frac{x_{0,n} + a_{1,n}}{1 + a_{1,n} + a_{2,n}}, \quad \forall n \in \{1, \ldots, N\}. \quad (4)
\]

**Motivation and interpretation for this rule**

1. If \(x_n(t)\) is seen as the probability of agent \(n\) picking the product of Firm 1, \(\phi(\cdot)\) corresponds to a Bayesian update rule on the opinion,

2. \(a_{i,n}\) is the increase in the odds of agent \(n\) choosing Firm \(i\),

3. Nice properties like symmetry, asymptotic limits etc.
Revenue per firm: $\mathcal{T} = \{0\}$
Agent influential power

The AIP of Agent $n$ is given by $\rho_n > 0$. When the profits are an integral of $x(t)$, the calculation of $\rho$ is given in [Varma et al, CDC 2017].

The revenue functions are taken as

$$u_1(x_0, a_1, a_2) := \rho \Phi(x_0, a_1, a_2) - \lambda_1 1 \top_N a_1,$$

(5)

$$u_2(x_0, a_1, a_2) := \rho \Phi(1 - x_0, n, a_2, n, a_1, n) - \lambda_2 1 \top_N a_2.$$

(6)

where $\lambda_i \geq 0$ is the advertising efficiency or pricing factor for Firm $i$. 
Static game formulation: \( \mathcal{T} = \{0\} \)

The strategic form of the *static game* of interest therefore writes as:

\[
G = (\{1, 2\}, \{\mathcal{A}_1, \mathcal{A}_2\}, \{u_1, u_2\}),
\]

where:

- \(\{1, 2\}\) is the set of players (i.e., Firms 1 and 2);
- \(\mathcal{A}_i\) defined in (1) is the set of pure actions for Player \(i\);
- \(u_i\) as defined per (5) (6) is the utility function for Firm \(i\).
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**Definition (Pure NE)**

A strategy profile $(a_1^*, a_2^*) \in A_1 \times A_2$ is a pure NE for $G$ for a given $x_0$ if $\forall i \in \{1, 2\}$,

$$\forall a_i \in A_i, \quad u_i(x_0, a_i^*, a_{-i}^*) \geq u_i(x_0, a_i, a_{-i}^*). \quad (8)$$
Theorem

The game $G$ has a pure and unique NE.
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Main steps of the proof:

1. Action space $A_i$ is a convex and compact set.
2. We show the utility function is concave w.r.t actions, enabling us to use the existence theorem in [Rosenthal et al 1965].
3. We show that the property of diagonally strict concavity in [Rosenthal et al 1965] is satisfied for uniqueness.
Let $\beta_i(x_{0;i}, a_{-i})$ be the BR. Then, at the NE

$$a_1^* \in \beta_1(x_{0;1}, a_2^*), \; a_2^* \in \beta_2(x_{0;1}, a_1^*)$$  \hspace{1cm} (9)
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**Proposition**

*The best-response functions are given by*

$$\beta_{i,n}(x_{0;i}, a_{-i}) = \min\{b_i, \max\{0, \alpha_{i,n}(x_{0;i}, a_{-i})\}\} \tag{10}$$

$$\alpha_{i,n}(x_{0;i}, a_{-i}) = \sqrt{\frac{\rho_n(x_{0,n;-i} + a_{-i,n})}{\mu_0;i + \lambda_i}} - 1 - a_{-i,n} \tag{11}$$

*for all $n \in \mathcal{V}$, and $\mu_0 \in \mathbb{R}_{\geq 0}$ is such that*

$$\sum_{n=1}^{N} \beta_{i,n}(x_{0;i}, a_{-i}) \leq B_i, \mu_{0;i}(\sum_{n} \beta_{i,n}(x_{0;i}, a_{-i}) - B_i) = 0 \tag{12}$$
Proposition

For each $n \in \mathcal{V}$, the NE $(a_{1,n}^*, a_{2,n}^*)$ is given by

- $(y, 0)$ (or $(0, y)$) if $\exists y \in [0, b_1]$ (or $[0, b_2]$ respectively) such that (9) is satisfied by one of these pairs,

- $(a_{1,n}^*, n, a_{2,n}^*) \in (0, b_1) \times (0, b_2)$ and is given by (13) where

$$k_i = \lambda_i + \mu_0; i$$

and $\mu_0; i$ is a common constant for all $n \in \mathcal{V}$ given by (12).
Proposition

For each $n \in \mathcal{V}$, the NE $(a^*_1,n, a^*_2,n)$ is given by

- $(y, 0)$ (or $(0, y)$) if $\exists y \in [0, b_1]$ (or $[0, b_2]$ respectively) such that (9) is satisfied by one of these pairs,
- or $(y, b_2)$ (or $(b_1, y)$) if $\exists y \in [0, b_1]$ (or $[0, b_2]$ respectively) such that (9) is satisfied by one of these pairs,
Characterization of the NE

Proposition

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- $(y, b_2)$ (or $(b_1, y)$) if $\exists y \in [0, b_1]$ (or $[0, b_2]$ respectively) such that (9) is satisfied by one of these pairs,

- $(a_{1,n}^*, a_{2,n}^*) \in (0, b_1) \times (0, b_2)$ and is given by

$$a_{i,n}^* = \left( \frac{k_i}{k_i + k_{-i}} \right)^2 k_{-i}\rho_n - x_{0,n;i}, \quad \text{(13)}$$

where $k_i = \frac{1}{\lambda_i + \mu_{0;i}}$ and $\mu_{0;i}$ is a common constant for all $n \in \mathcal{V}$ given by (12).
Uniform broadcasting allocation (UBA)

When firm $i$ uses UBA strategy, then

$$a_i^{\text{UBA}} := \min \left\{ b_i, \frac{B_i}{N} \right\}. \quad (14)$$
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$$a_i^{UBA} := \min \left\{ b_i, \frac{B_i}{N} \right\}. \quad (14)$$

The resulting difference in utility between the two strategies is referred to as the *gain of targeting* (GoT), and is measured as

$$\text{GoT} := \frac{u_1(x_0, \beta_1(a_2^{UBA}), a_2^{UBA}) - u_1(x_0, a_1^{UBA}, a_2^{UBA})}{u_1(x_0, a_1^{UBA}, a_2^{UBA})}. \quad (15)$$
For this simulation, we consider $N = 100$ with $\rho_n \in \{1, C\}$. $C$ being the AIP of leaders.
NE for a given graph and initial opinions

Figure 1: The sub-figure on top shows the AIP $\rho_n$ and initial opinion $x_n(0)$, the sub-figure on the bottom shows the $a_i^*$. 
Repeated campaigns: applying the OS-NE 16/22

Here, we take $N = 5$, $\lambda_1 = 2$, $\lambda_2 = 1$ and

$$\rho = (6.8910, 1.9202, 1.30631, 1.036, 3.7789)^T.$$
Campaigns instances given by $t_k$, $k \in \mathcal{K} \subset \mathbb{Z}$.

Net utilities given by

$$U_i = \sum_{k \in \mathcal{K}} u_i(x(t_k), a_1(k), a_2(k))$$  \hspace{1cm} (16)

Let

$$\overline{X}_n := \left\{ y \in \mathbb{R} : y > 1 - \frac{\lambda_1}{\rho_n}, y < \frac{\lambda_2}{\rho_n} \right\}$$  \hspace{1cm} (17)

and

$$\eta := \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$  \hspace{1cm} (18)
Theorem

Let $\rho_{\text{max}} := \min_{k \in \mathcal{K}} \max_{n \in \mathcal{N}} \rho_n(k)$. Assume Marketer $i$, $i \in \{1, 2\}$, implements the marketing strategy $\sigma^*_i$. Assume the graph associated with the matrix $L$ to be strongly connected. Then the dynamical system (3) has at least one (system) equilibrium which verifies the following:

- If $\frac{\rho_{\text{max}}}{\lambda_1 + \lambda_2} > 1$, then $x_n = \eta$, $n \in \mathcal{N}$, is the unique equilibrium.
- If $\frac{\rho_{\text{max}}}{\lambda_1 + \lambda_2} \leq 1$, then any $x_n \in \overline{X}_{\text{max}}$, $n \in \mathcal{N}$ is an equilibrium, where $\overline{X}_{\text{max}}$ is defined by replacing $\rho_n$ with $\rho_{\text{max}}$ in (17).
Coopetition plan

- Apply the OS NE strategy for $K_1$ stages
- For all $k > K_1$, both players agree to do $a_1(k) = a_2(k) = 0$.

Coopetition plan is sustainable if it pareto-dominates the OS NE strategy.
Trivially, if $x(t_k) = \eta 1_N$ at some $k$, applying 0 will pareto-dominate the NE.
Here, we take $N = 5$, $\lambda_1 = 2$, $\lambda_2 = 1$ and

$$\rho = (6.8910, 1.9202, 1.30631.1036, 3.7789)^T.$$
Repeated campaigns: applying the OS-NE 21/22

Here, we take $N = 5$, $\lambda_1 = 3$, $\lambda_2 = 1.5$ and

$$\rho = (6.8910, 1.9202, 1.30631, 1.036, 3.7789)^T.$$
Future directions

- imperfect/noisy information on $\rho$ (or $L$) and $x(0)$.
- continuous control

Thanks for your attention